Announcements

1) Midterm due Thursday,
   8 PM

2) Guest lecturer for
   tomorrow: Tom Fiore

Recall: linear transformation

Equivalent Terminology

"Linear transformation"

is the same as

"Linear map" or

"Linear operator."

Example 1 : (integration)

Consider $V = C(\mathbb{R})$

as a vector space over $\mathbb{R}$.

Define $T: C(\mathbb{R}) \to \mathbb{R}$

by

$$\boxed{T(f) = \int_0^1 f(x)\,dx}$$

$\forall f \in C(\mathbb{R})$

The integral is the usual Riemann integral.

T is linear.

1) Let $f, g \in C(\mathbb{R})$.

$$T(f+g) = \int_0^1 (f(x)+g(x))\,dx$$

property of the integral

$$= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$$

$$= T(f) + T(g)$$

2) If $f \in C(\mathbb{R})$, $\alpha \in \mathbb{R}$,

$$T(\alpha f) = \int_0^1 \alpha \cdot f(x)\,dx$$

$$\xrightarrow{\hspace{2em}} \quad = \alpha \int_0^1 f(x)\,dx$$

property of the integral

$$= \alpha T(f).$$

Therefore T is linear.

If V is a vector space over $\mathbb{F}$ and $T: V \to \mathbb{F}$ is linear, we say T is a linear functional on V. The set of all such maps is called the dual of V and denoted by V^*.

Example 2: (the shift)

Let $V = \ell_2(\mathbb{N})$ (over $\mathbb{C}$)

$$= \left\{ (\alpha_n)_{n=1}^{\infty} \;\middle|\; \alpha_n \in \mathbb{C} \text{ \& } \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \right\},$$

Define $T: \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$

by $\boxed{T\left((\alpha_n)_{n=1}^{\infty}\right) = (\beta_n)_{n=1}^{\infty}}$

where $\boxed{\beta_1 = 0,\ \beta_n = \alpha_{n-1}\ \forall\, n \geq 2.}$

T shifts sequences one spot to the right and puts a zero in the first coordinate.

T is linear.

The proof is easy.

Definition: (isomorphism)

Let V and W be vector spaces over $\mathbb{F}$. An **isomorphism** from V to W is a **linear** map $T: V \to W$ that is **bijective**.

Example 3: $\mathbb{R}^2$ vs $\mathbb{C}$

Final Battle!

Consider both as vector spaces over $\mathbb{R}$. Define $T: \mathbb{R}^2 \to \mathbb{C}$ by

$$\boxed{T((x,y)) = x + iy}$$

$\forall\ x, y \in \mathbb{R}$.

T is an isomorphism.

$\underline{T \quad \text{linear}}$

1) Let $(x,y), (a,b) \in \mathbb{R}^2$.

$$T((x,y)+(a,b))$$

$$= T((x+a, y+b))$$

$$= (x+a) + i(y+b)$$

$$= (x+iy) + (a+ib)$$

$$= T((x,y)) + T((a,b))$$

2) Let $(x,y) \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$.

Then $T(\alpha(x,y))$

$$= T((\alpha x, \alpha y))$$

$$= \alpha x + i(\alpha y)$$

$$= \alpha(x + iy)$$

$$= \alpha T((x,y))$$

T is linear.

3) Injectivity:

Suppose $T((x,y)) = T((a,b))$.

Then by linearity,

$$0 = T((x,y)) - T((a,b))$$

$$\text{linearity of } T \left\{ \begin{aligned} &= T((x,y)) + T((-a,-b)) \\ &= T((x-a,\, y-b)). \end{aligned} \right.$$

$$= (x-a) + i(y-b).$$

Then $x=a$ and $y=b$.

So T is injective.

4) Surjectivity:

If $x+iy \in \mathbb{C}$, then

$T((x,y)) = x+iy$.

T is bijective.

Hence, T is an isomorphism.

Note

1) If $T: V \to W$ is an isomorphism, we say V and W are isomorphic (as vectur spaces).

2) A consequence of the previous proof (liberally interpreted) is that if $T: V \to W$ is linear, then T is injective if and only if $T(x) = 0_W$

$\Rightarrow x = 0_V \quad \forall x \in V.$

Example 4 (c_{00} and $\mathcal{P}$)

Let $p(x) = \sum_{k=0}^{n} a_k x^k$

be a polynomial with real coefficients and let c_{00} denote all real sequences $(a_k)_{k=1}^{\infty}$ such that $\exists$ $N \in \mathbb{N}$, $a_k = 0$ $\forall$ $k \geq N$.

If $\mathcal{P}$ is the vector space of all polynomials with real coefficients (over $\mathbb{R}$), define

$T : \mathcal{P} \to C_{00}$ by

$$\boxed{\begin{aligned} &T\left(\sum_{k=0}^{n} \alpha_k x^k\right) \\ &= (\alpha_0, \alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \end{aligned}}$$

Then T is an isomorphism – Check!

Theorem: (finite dimensions)

Let V be a finite-dimensional vector space over $\mathbb{F}$. If $n = \dim(V)$, then V is isomorphic to $\mathbb{F}^n$.

Proof: Since $\dim(V) = n < \infty$,

$\exists$ a basis

$\{x_1, x_2, \ldots, x_n\}$

for V over $\mathbb{F}$.

Let f_i, $1 \leq i \leq n$, denote

the element in $\mathbb{F}^n$ with

i^{th} coordinate equal to $1_{\mathbb{F}}$

and all other coordinates equal to $0_{\mathbb{F}}$.

If $x \in V$, write

$$x = \sum_{k=1}^{n} \alpha_k x_k \text{ for } \alpha_k \in \mathbb{F},$$

$1 \le k \le n$. Define

$T : V \to \mathbb{F}^n$ by

$$\boxed{T(x) = \sum_{k=1}^{n} \alpha_k f_k}.$$

T is an isomorphism -

you need to check that T is well-defined, which is equivalent to showing that if $x = \sum_{k=1}^{n} \alpha_k x_k$,

the coefficients are uniquely determined.

So let

$$x = \sum_{k=1}^{n} \alpha_k x_k = \sum_{k=1}^{n} \beta_k x_k$$

for some $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{F}$.

Then

$$0_{\mathbb{F}} = x - x = \sum_{k=1}^{n} (\alpha_k - \beta_k) x_k$$

By linear independence of $\{x_k\}_{k=1}^{n}$,

$\alpha_k - \beta_k = 0_{\mathbb{F}} \ \forall\ 1 \leq k \leq n$,

so $\alpha_k = \beta_k \ \forall\ 1 \leq k \leq n$

and T is well-defined.

That T is linear is a formality:

Let $x = \sum_{k=1}^{n} \alpha_k x_k$,

$y = \sum_{k=1}^{n} \beta_k x_k$ for $\alpha_k, \beta_k \in \mathbb{F}$

$\forall\ 1 \leq k \leq n$.

Then

$$T(x+y) = T\left(\sum_{k=1}^{n}(\alpha_k+\beta_k)x_k\right)$$

$$= \sum_{k=1}^{n}(\alpha_k+\beta_k)f_k$$

$$= \sum_{k=1}^{n}\alpha_k f_k + \sum_{k=1}^{n}\beta_k f_k$$

$$= T(x) + T(y)$$

If $\alpha \in \mathbb{F}$,

$$T(\alpha x) = T\left(\sum_{k=1}^{n} \alpha \alpha_k x_k\right)$$

$$= \sum_{k=1}^{n} \alpha \alpha_k f_k$$

$$= \alpha \sum_{k=1}^{n} \alpha_k f_k$$

$$= \alpha T(x)$$

So T is linear.

Now suppose

$T(x) = T(y)$.

Then

$$\sum_{k=1}^{n} \alpha_k f_k = \sum_{k=1}^{n} \beta_k f_k$$

$$\Rightarrow \sum_{k=1}^{n} (\alpha_k - \beta_k) f_k = 0_{\mathbb{F}^n}$$

$\Rightarrow \alpha_k = \beta_k \quad \forall\, 1 \leq k \leq n$

by linear independence of $\{f_k\}_{k=1}^{n}$

Hence, T is injective.

Now if $z = \sum_{k=1}^{n} \alpha_k f_k \in \mathbb{F}^n$,

$$T\left(\sum_{k=1}^{n} \alpha_k x_k\right) = z.$$

This shows T is surjective. Combining everything we have proved, T is an isomorphism from V to $\mathbb{F}^n$. $\square$